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Application of Bond-Moving Renormalization-Group Approach to Fractal Lattices

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We investigate the application of the Migdal–Kadanoff bond-moving renormalization group (RG) approach to fractal lattices. We find the following two results: first, for inhomogeneous interaction lattice models, bond moving involving inequivalent bonds is unsuitable because it violates the condition $\langle \Delta \rangle = 0$ (Δ is the perturbation potential resulting from moving the bonds); second, the condition $\langle \Delta \rangle = 0$ does not uniquely determine the way to move bonds; different choices of bond moving yield different RG recursion relations and corresponding fixed points, which makes the conclusions concerning the phase transition quite uncertain.

KEY WORDS: Bond-moving renormalization group; phase transition; Ising model.

1. INTRODUCTION

The bond-moving renormalization group (RG) approach proposed by Migdal and Kadanoff is a significant real-space RG approach.[1,2] It is a lower bound approximation based on a variational principle which makes the sum of the partition function calculable and optimizes the result. It has been extensively applied to equilibrium systems on translational symmetry lattices, and in the past 10 years it has also been applied to fractal lattices with self-similar symmetry.

Kadanoff[2] has analyzed the limitations and weaknesses of applying this approach. Besides being inaccurate, a usual defect is that when it is applied to d-dimensional ($d \geqslant 2$) translational lattice systems with isotropic

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interactions, the isotropic feature is lost when the RG procedure is performed in different directions. The reason is that bond moving is always combined with a decimation procedure in the renormalization process; since bond-moving and decimation procedures are noncommutative, this leads to the occurrence of anisotropy. Usually, to restore isotropy, a symmetrized procedure has to be introduced.[3]

Since the 1980s much attention has been given to fractal physics. Gefen *et al.*[4–6] studied phase transitions of Ising and Potts models on the Koch curve, the Sierpinski gasket, and Sierpinski carpets (SC). They introduced two interaction parameters K and K_W and used the bond-moving RG approach to investigate phase transitions on Sierpinski carpets with infinite ramification order.[6] This approach has since been used to study critical dynamics.[7] However, in a recent study we found that those investigations neglected some essential points of the bond-moving RG approach. In this paper we point out that moving potential terms from bonds to inequivalent bonds is unsuitable, because it violates the condition (5) given below. We give a detailed interpretation of this in Section 3.

We will also see below that Eq. (5) allows us an extensive degree of freedom in choosing additional perturbation potentials Δ. Under the constraint of Eq. (5) we may choose different ways of bond moving and then obtain different recursion relations of the renormalization group. The fixed points of these relations may be very different and this makes the conclusions concerning phase transitions quite uncertain. We treat this in Section 4.

2. A BRIEF REVIEW

First, we give a brief review of the bond-moving RG approach. Following Kadanoff's derivation,[2,3] suppose we have the original Hamiltonian $H(\sigma)$, and under a scale transformation $a \to \lambda a$, carry out a transformation $T(\mu, \sigma)$ from the old variables σ to new ones μ, resulting in a new renormalized Hamiltonian $H'[\mu]$

$$H'(\mu) = \ln \mathrm{Tr}_\sigma \exp[T(\mu, \sigma) + H(\sigma)], \tag{1}$$

where $T(\mu, \sigma)$ is constrained by the condition

$$\mathrm{Tr}_\mu \exp T(\mu, \sigma) = 1 \tag{2}$$

so that the partition function of the system is left invariant:

$$Z[H(\sigma)] = Z[H'(\mu)] \tag{3}$$

We now introduce an approximate renormalized Hamiltonian

$$H'_A(\mu) = \ln \mathrm{Tr}_\sigma \exp[T(\mu, \sigma) + H(\sigma) + \Delta(\mu, \sigma)] \tag{4}$$

where $\Delta(\mu, \sigma)$ is chosen in terms of a variational method so that the sum is calculable. Kadanoff found that under the conditions

$$\begin{aligned} &\text{(a)} \quad \mathrm{Tr}_\sigma \exp[T(\mu, \sigma) + H(\sigma)] \text{ is a sum with positive-semidefine weights} \\ &\text{(b)} \quad \Delta(\mu, \sigma) \text{ is real and small} \\ &\text{(c)} \quad \mathrm{Tr}_\mu \mathrm{Tr}_\sigma \exp[T(\mu, \sigma) + H(\sigma)]\, \Delta(\mu, \sigma) = 0; \qquad \text{i.e.,} \quad \langle \Delta \rangle = 0 \end{aligned} \tag{5}$$

where $\langle \Delta \rangle$ represents a statistical average of the function Δ, the free energy of the system satisfies the following relation:

$$F[H'_A] \leqslant F[H'] = F[H] \tag{6}$$

A simple choice for Δ is

$$\Delta(\mu, \sigma) = \sum_{i,\alpha} c_i^\alpha a_i^\alpha(\mu, \sigma) \tag{7}$$

where $a_i^\alpha(\mu, \sigma)$ denotes a local coupling term (e.g., $\sigma_i \sigma_{i+1}$) and c_i^α is independent of μ and σ, and we impose the condition that for all α

$$\sum_i c_i^\alpha = 0 \tag{8}$$

For this choice, we find that Eq. (5) certainly holds. The condition (8) has a simple geometrical interpretation: The variational principle allows us to move potential terms (coupling terms) from a set of bonds in the lattice to equivalent bonds, but not to increase or decrease the total amount of any type of bond. This is the theoretical basis of the bond-moving RG, which implies that "bond moving" refers specifically to moving with $\langle \Delta \rangle = 0$.

3. INHOMOGENEOUS INTERACTION MODEL ON FRACTAL LATTICE

Let us start from an inhomogeneous interaction Ising model on the Sierpinski carpet; we write the model Hamiltonian as[6]

$$\frac{-H}{k_B T} = K \sum_{n.n.} \sigma_i \sigma_j + K_w \sum_{n.n.} \sigma'_i \sigma'_j \tag{9}$$

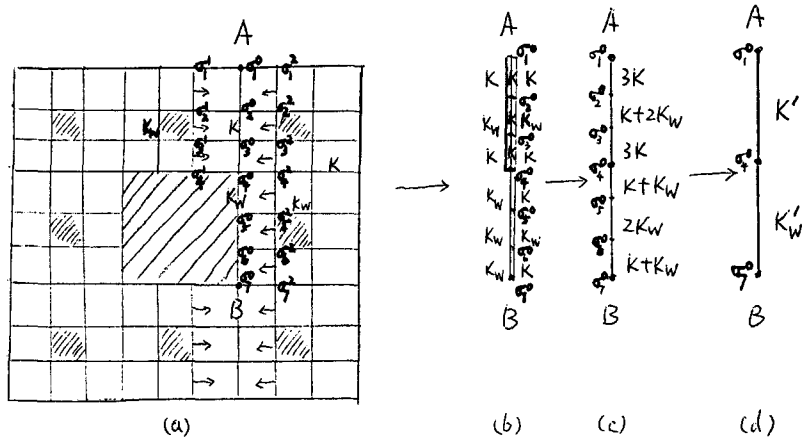

Fig. 1. A scheme for a bond-moving renormalization-group transformation on the Sierpinski carpet. (a) A K-bond and a K_W-bond; arrows denote directions of movement. (b, c) Results moving bonds along line AB. (d) Renormalized K' and K'_W bonds via decimation procedure.

where $K = J/K_B T$ (K_B is the Boltzmann constant and T is the temperature) denotes the interaction via a bond which separates two noneliminated subsquares, and $K_W = J_W/K_B T$ is the coupling via a bond which borders an eliminated subsquare. We have $\sigma_i \equiv \sigma'_i$ at the intersection of a K-bond and a K_W-bond. To work out the renormalization group transformation, a bond-moving RG scheme has been performed[6] (see Fig. 1) and a set of coupled RG recursion relations obtained.

It is worth noting that in the bond-moving scheme shown in Fig. 1 there are two kinds of bond moving, i.e., moves of a K (K_W) bond from one place to a K (K_W) bond at another place, and moves of a K_W (K) bond from one place to a K (K_W) bond at another place (see that below); the latter is bond moving between inequivalent bonds. We see, for example, that two K-bonds between spins σ_1^1 and σ_2^1 and between spins σ_1^2 and σ_2^2 are moved to between spins σ_1^0 and σ_2^0, where there was a K-bond originally (bond moving between equivalent bonds), while two K_W-bonds between spins σ_2^1 and σ_3^1 and between spins σ_2^2 and σ_3^2 are moved to between spins σ_2^0 and σ_3^0, where there was a K-bond originally (bond moving between inequivalent bonds). Corresponding to these bond movings we can write the corresponding terms included in an additional perturbation function Δ; they are $(2K\sigma_1^0\sigma_2^0 - K\sigma_1^1\sigma_2^1 - K\sigma_1^2\sigma_2^2)$ and $(2K_W\sigma_2^0\sigma_3^0 - K_W\sigma_2^1\sigma_3^1 - K_W\sigma_2^2\sigma_3^2)$. Immediately we can calculate their contribution to $\langle \Delta \rangle$ as follows:

$$\sum_{\sigma} e^{-H/K_B T}(2K\sigma_1^0\sigma_2^0 - K\sigma_1^1\sigma_2^1 - K\sigma_1^2\sigma_2^2) = 0 \tag{10}$$

and

$$\sum_{\sigma} e^{-H/K_B T}(2K_W\sigma_2^0\sigma_3^0 - K_W\sigma_2^1\sigma_3^1 - K_W\sigma_2^2\sigma_3^2) \neq 0 \tag{11}$$

They represent the contributions of two typical bond-moving terms. The difference in the results (10) and (11) comes from the fact that there is exchange symmetry between equivalent bonds, while there is none between inequivalent bonds. In more detail, there are terms

$$(K\sigma_1^0\sigma_2^0 + K\sigma_1^1\sigma_2^1 + K\sigma_1^2\sigma_2^2) + (K\sigma_2^0\sigma_3^0 + K_W\sigma_2^1\sigma_3^1 + K_W\sigma_2^2\sigma_3^2)$$

in the Hamiltonian H; obviously, in the last parentheses the exchange between K and K_W is asymmetric. Consequently, we obtain that in an inhomogeneous interaction lattice model, bond moving between inequivalent bonds must lead to $\langle \Delta \rangle \neq 0$, i.e., violation of the condition $\langle \Delta \rangle = 0$. Therefore bond moving between inequivalent bonds is unsuitable within the scheme discussed by Kadanoff.

4. DIFFERENT WAYS OF MOVING BONDS

As we mentioned above, the condition $\langle \Delta \rangle = 0$ can be simply realized in the bond-moving scheme. However, this condition does not uniquely determine the way to move bonds, because it leaves extensive room for the choice of the function Δ. In other words, there are various choices for the specific form of bond moving even for a homogeneous interaction lattice model. Moreover, we may move not only integer numbers of bonds, but also fractional numbers of bonds. That makes the results of the renormalization group quite different and may even lead to completely opposite conclusions concerning the phase transition of the system.

Let us investigate the phase transition of the Ising model with only nearest neighbor interactions on the Sierpinski gasket, which has also been studied by Gefen *et al.*[5] We find that when we choose different ways of moving bonds to treat the same object we end up with different or even contradictory results.

In Fig. 2 we perform Ising bond moving: the bond connecting spins σ_3 and σ_5 is moved to between spins σ_1 and σ_2 with strength αK and between spins σ_2 and σ_4 with strength $(1-\alpha)K$; α can vary continuously within $0 \leqslant \alpha \leqslant 1$. The bonds between spins σ_2 and σ_3 and between spins σ_2 and σ_5

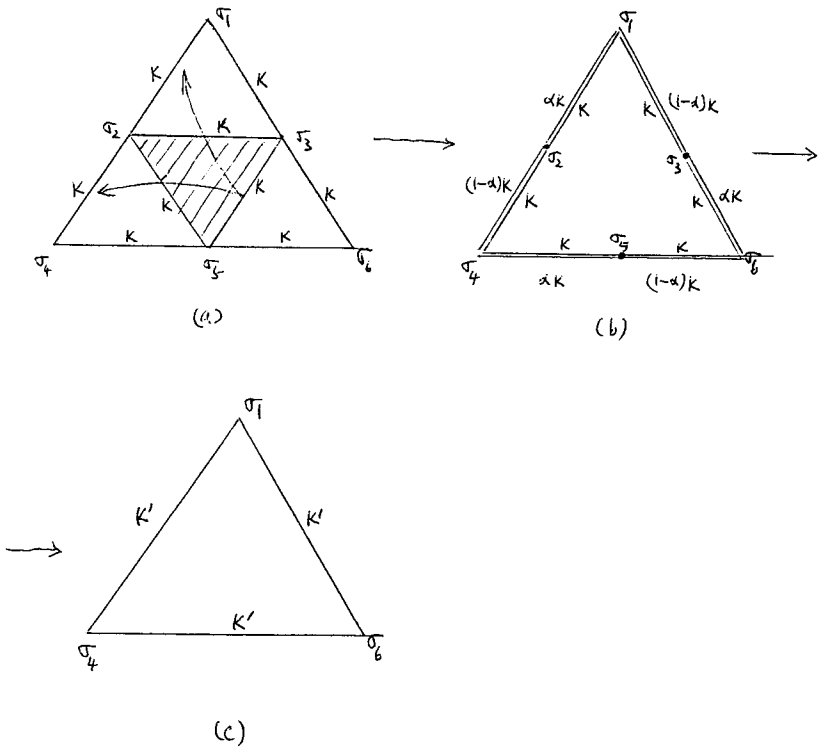

Fig. 2. A scheme for a bond-moving renormalization-group transformation on the Sierpinski gasket. (a) For example, a K-bond between σ_3 and σ_5 is moved to between σ_1 and σ_2 with strength αK and between σ_2 and σ_4 with strength $(1-\alpha)K$; arrows denote directions of movement. (b) Result of bond moving. (c) Renormalized K'-bond via decimation procedure.

are also moved similarly. The additional perturbation function Δ is then written as

$$\begin{aligned}\Delta = &[\alpha K\sigma_1\sigma_2 + (1-\alpha)\,K\sigma_2\sigma_4 - K\sigma_3\sigma_5] \\ &+ [\alpha K\sigma_3\sigma_6 + (1-\alpha)\,K\sigma_1\sigma_3 - K\sigma_2\sigma_5] \\ &+ [\alpha K\sigma_4\sigma_5 + (1-\alpha)\,K\sigma_5\sigma_6 - K\sigma_2\sigma_3]\end{aligned} \tag{12}$$

It is easy to prove that $\langle \Delta \rangle = 0$ is satisfied.

After following a decimation procedure, we obtain the RG recursion relation as

$$\tanh K' = \tanh(2-\alpha)K \tanh(1+\alpha)K \tag{13}$$

To determine the critical point of the phase transition, we have to find fixed points of the RG recursion relation. In the following we list all the fixed points for special choices of α:

$$\begin{cases} K^* = 0, \infty, & \alpha = 0, \\ K^* = 0, 0.675975, \infty, & \alpha = 1/2 \\ K^* = 0, 0.75109, \infty, & \alpha = 3/4 \\ K^* = 0, 1.73201, \infty, & \alpha = 0.99 \\ K^* = 0, \infty, & \alpha = 1 \end{cases} \tag{14}$$

Obviously, when $\alpha = 0, 1$ only trivial fixed points exist and thus no finite-temperature phase transition occurs in the Sierpinski gasket, which agrees with the known exact result.[5] However, when $\alpha \neq 0, 1$ nontrivial fixed points occur, which implies the existence of a finite-temperature phase transition, with a critical temperature which depends on the value of α. From the above we come to the following conclusion: the existence of a phase transition of the Ising model on the Sierpinski gasket lattice depends on the choice of how the bonds are moved. Obviously this is inconsistent with real phenomena in nature. However, it is the logical consequence of the bond-moving RG approach.

As we mentioned before, the bond-moving approach is a lower bound approximation of the free energy; thus a "best" choice of the perturbation potential Δ is to make the approximate free energy F'_Δ a maximum by varying the free parameter α. We have found that the best one corresponds to $\alpha = 1/2$, which implies the occurrence of a phase transition at a finite temperature, inconsistent with the exact RG result.[5]

In the following we perform an unusual way of moving bonds on the square lattice, which derives a quite different RG recursion relation. Figure 3 shows such a bond-moving RG process. The K-bond between σ_1 and σ_2 and that between σ_2 and σ_3 are both moved to between σ_7 and σ_8, for example. The RG recursion relations are given by

$$\begin{aligned} \tanh K'_X &= \tanh K \tanh 3K \\ \tanh K'_Y &= \tanh[2(\tanh^{-1} \tanh^2 K] \end{aligned} \tag{15}$$

in contrast to the results of ref. 3. However, it is easy to prove that the "best" choice of bond moving is the homogeneous scheme used in ref. 3.

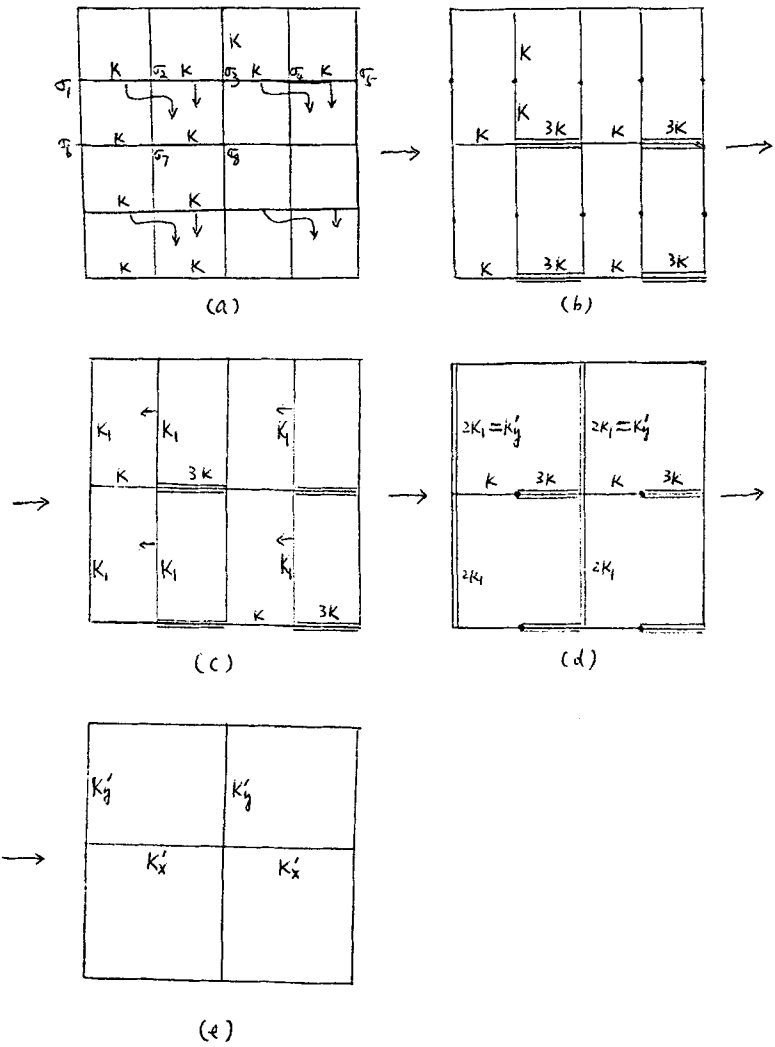

Fig. 3. An unusual scheme for a bond-moving renormalization-group transformation on the square lattice. (a) For example, K-bonds between σ_1 and σ_2 and between σ_2 and σ_3 are moved to between σ_7 and σ_8; arrows denote directions of movement. (b) Result of bond-moving. (c) Resulting K_1-bond via decimation, then movement of K_1-bond. (d) Renormalized K'_y-bonds. (e) Renormalized K'_x-bonds.

5. CONCLUSION AND DISCUSSION

We have investigated in detail the application of the bond-moving renormalization group to fractal lattices. Our main findings indicate two aspects which have been neglected before:

First, for an inhomogeneous interaction lattice model, only bond moving between equivalent bonds is suitable; otherwise, the process will violate the condition $\langle \Delta \rangle = 0$. In fact, for systems with inequivalent bonds such as the Sierpinski carpet, the previously used RG bond-moving scheme does not obey Kadanoff's condition $\langle \Delta \rangle = 0$.

Second, under the constraint condition $\langle \Delta \rangle = 0$, the choice of bond-moving scheme is arbitrary; in other words, we may have different ways to move bonds without violating $\langle \Delta \rangle = 0$. Because of this, we may find completely different RG recursion relations, depending on the different choices of bond-moving schemes, which makes any conclusion concerning the phase transition quite uncertain. Therefore the scheme is generally not very reliable.

Furthermore, we may in principle choose the additional perturbation function Δ which may not correspond to any simple geometrical interpretation, as long as $\langle \Delta \rangle = 0$ holds and the sum of the partition function is calculable. Such a choice seems very complicated and difficult to find; we do not know if there is such a choice.

Kadanoff already pointed out some difficulties and problems with the bond-moving RG approach and explained their causes. However, when it is applied to fractal lattices, some new difficulties emerge, as mentioned here. Of course, our findings are also applicable to translational symmetry lattices and other network structures.

Finally, we would like to stress that the bond-moving RG approach is not a good one; its results and conclusions have only limited value at best.

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